THE UNIVERSITY OF CHICAGO

MINIMA OF FUNCTIONS OF SEVERAL VARIABLES WITH INEQUALITIES AS SIDE CONDITIONS

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MINIMA OF FUNCTIONS OF SEVERAL VARIABLES WITH INEQUALITIES AS SIDE CONDITIONS

1. Introduction. The problem of determining necessary conditions and sufficient conditions for a relative minimum of a function $f(x_1,x_2,\ldots,x_n)$ in the class of points $x = (x_1,x_2,\ldots,x_n)$ satisfying the equations $g_\alpha(x) = 0$ ($\alpha = 1,2,\ldots,m$), where the functions f and g_α have continuous derivatives of at least the second order, has been satisfactorily treated [1]*. This paper proposes to take up the corresponding problem in the class of points x satisfying the inequalities

$$g_\alpha(x) \geqq 0 \qquad (\alpha = 1,2,\ldots,m), \tag{1}$$

where m may be less than, equal to, or greater than n.

We shall be interested in a minimizing point x^o at which all the functions g_α vanish. The reason we limit our attention to this case is that if $f(x^o)$ = minimum and, say, $g_1(x^o) > 0$ then by continuity $g_1(x) \geqq 0$ for all x sufficiently close to x^o and hence the condition $g_1(x) \geqq 0$ puts no restriction on the problem so far as the theory of relative minima is concerned. Henceforth in this paper whenever we state "$f(x^o)$ is a minimum" or "x^o is a minimizing point" we assume that $g_\alpha(x^o) = 0$ for every α.

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*Numbers in brackets refer to the list of references at the end of the paper.

We shall not limit ourselves to the case when f and g_α are of class C". In Sections 3 and 4 we consider the minimum problem under the assumption that the functions f and g_α are merely of class C' near a point $x = x^0$. However in Sections 5 and 6 we do restrict attention to the case when the functions are of class C". Section 2 will deal with some properties of linear inequalities.

We shall have occasion to use all the results in the first part of Bliss's paper [1] and for convenience we list them here. They are concerned with the problem of minimizing f(x) in the class of points x satisfying equations

$$h_\alpha(x) = 0 \qquad (\alpha = 1,2,\ldots,m < n),$$

and may be compared with the results obtained in this paper. One needs only continuous first derivatives for Theorem 1:1 and continuous second derivatives for the other theorems.

THEOREM 1:1. <u>A first necessary condition for</u> $f(x^0)$ <u>to be a minimum is that there exist constants</u> χ_0, χ_α <u>not all zero such that the derivatives</u> H_{x_i} <u>of the function</u>

$$H = \chi_0 f + \chi_\alpha h_\alpha$$

<u>all vanish at</u> x^0.

LEMMA 1:1. <u>If</u> $\| h_{\alpha x_i}(x^0) \|$ <u>has rank</u> m, <u>then for every set of constants</u> η_i $(i = 1,2,\ldots,n)$ <u>satisfying the equations</u>

$$h_{\alpha x_i}(x^0)\, \eta_i = 0$$

<u>there exists a curve</u> $x_i(t)$ <u>having continuous second derivatives near</u> $t = 0$, <u>satisfying the equations</u> $h_\alpha[x(t)] = 0$, <u>and such that</u>

$$x_i(0) = x_i^0, \qquad x_i'(0) = \eta_i .$$

THEOREM 1:2. *If* $\| h_{\alpha x_i}(x^o) \|$ *has rank* m *and* $f(x^o)$ *is a minimum then the condition*

$$H_{x_i x_k}(x^o)\eta_i \eta_k \geqq 0$$

must hold for every set η_i *satisfying* $h_{\alpha x_i}(x^o)\eta_i = 0$, *where* $H = f + \chi_\alpha h_\alpha$ *is the function formed with the unique set of multipliers* $\chi_o = 1$, χ_α *belonging to* x^o.

Our final excerpt from Bliss's paper is a sufficiency theorem.

THEOREM 1:3. *If a point* x^o *has a set of multipliers* $\chi_o = 1$, χ_α *for which the function* $H = f + \chi_\alpha h_\alpha$ *satisfies the conditions*

$$H_{x_i}(x^o) = 0, \qquad H_{x_i x_k}(x^o)\eta_i \eta_k > 0$$

for all sets η_i *satisfying the equations*

$$h_{\alpha x_i}(x^o)\eta_i = 0,$$

then $f(x^o)$ *is a minimum*.

2. *Preliminary theorems on linear inequalities*. To introduce the important theorem which is about to follow we consider the system of linear inequalities

$$\text{(2)} \qquad \begin{aligned} L_1 &\equiv A_{11}u_1 + A_{12}u_2 + \dots + A_{1n}u_n \geqq 0 \\ L_2 &\equiv A_{21}u_1 + A_{22}u_2 + \dots + A_{2n}u_n \geqq 0 \\ &\dots \qquad \dots \qquad \dots \qquad \dots \\ L_m &\equiv A_{m1}u_1 + A_{m2}u_2 + \dots + A_{mn}u_n \geqq 0, \end{aligned}$$

in which the A's are real constants. If for every solution u of (2) the inequality

$$\text{(2')} \qquad \Phi \equiv A_1u_1 + A_2u_2 + \dots + A_nu_n \geqq 0$$

is satisfied, then the inequality (2') is called a consequence of the system of inequalities (2). Farkas, in his paper [4], proved the following theorem. (See also Corollary 1, p. 47 of Dines and McCoy [3]).

THEOREM 2:1. *If (2') is a consequence of (2) then there exist non-negative constants C_α such that*

$$\Phi \equiv C_1L_1 + C_2L_2 + \dots + C_mL_m.$$

The solution $(u_1,u_2,\dots,u_n) = (0,0,\dots,0)$ will be called a trivial solution of (2). We note that the theorem does not assume that there necessarily exists a non-trivial solution of (2).

We make an inductive proof. If $n = 1$ the conclusion is readily verified. We suppose the theorem true for $n-1$ variables $u_1,u_2,\dots,u_{n-1}$ and make the proof for n variables. If $\Phi \equiv 0$ then the conclusion is obvious. Hence we assume some A_i is different from zero and, for convenience, let $A_n \neq 0$. Solving $\Phi = A_1u_1 + A_2u_2 + \dots + A_nu_n$ for u_n we obtain $u_n = \frac{\Phi}{A_n} - \frac{1}{A_n}\left[A_1u_1 + \dots + A_{n-1}u_{n-1}\right]$ which we substitute in L_i. The result is

$$L_i \equiv A_{i1}u_1 + \dots + A_{in-1}u_{n-1} + \frac{A_{in}}{A_n}\Phi - \frac{A_{in}}{A_n}\left[A_1u_1 + \dots + A_{n-1}u_{n-1}\right] \geqq 0. \tag{3}$$

If the coefficient of Φ is different from zero divide both sides of (3) by $\left|\frac{A_{in}}{A_n}\right|$ and obtain an inequality $\bar{L}_i \geqq 0$ in which the coefficient of Φ is ± 1. Since $\bar{L}_i$ is a positive multiple of L_i we can replace the latter by $\bar{L}_i$ in (2). This we do and, to simplify notation, drop the bar over L_i. With this understanding the system (2) can be rewritten as

$$\begin{array}{llll} L_{i_1} \equiv \Phi + P_1 \geqq 0, & L_{i_2} \equiv \Phi + P_2 \geqq 0, & \dots, & L_{i_r} \equiv \Phi + P_r \geqq 0, \\ L_{j_1} \equiv -\Phi + N_1 \geqq 0, & L_{j_2} \equiv -\Phi + N_2 \geqq 0, & \dots, & L_{j_s} \equiv -\Phi + N_s \geqq 0, \\ L_{k_1} \equiv \quad Z_1 \geqq 0, & L_{k_2} \equiv \quad Z_2 \geqq 0, & \dots, & L_{k_t} \equiv \quad Z_t \geqq 0, \end{array} \tag{4}$$

where $r + s + t = m$ and the $P_1, P_2, \ldots, N_1, N_2, \ldots, Z_1, Z_2, \ldots$ are linear forms in $u_1, u_2, \ldots, u_{n-1}$. If we consider (4) as a system of inequalities with independent variables $u_1, \ldots, u_{n-1}, \Phi$ then from the fact that (2') is a consequence of (2) it follows that

$$\Phi \geqq 0 \tag{4'}$$

is a consequence of (4).

There is at least one linear form in $(u_1, u_2, \ldots, u_{n-1}, \Phi)$ of the type displayed in the first line of (4). For, if this were not the case then $(u_1, u_2, \ldots, u_{n-1}, \Phi) = (0, 0, \ldots, 0, -1)$ would be a solution of (4) in contradiction to (4'). We may also assume that no one of the P's is identically zero, since if for example $P_1 \equiv 0$ then $\Phi \equiv L_{i_1}$ and the conclusion would hold. By adding each inequality in the first line of (4) to each inequality in the second line we obtain

$$\begin{array}{lll}
L_{i_1} \equiv \Phi + P_1 \geqq 0, & \ldots, & L_{i_r} \equiv \Phi + P_r \geqq 0 \\
L_{i_1} + L_{j_1} \equiv P_1 + N_1 \geqq 0, & \ldots, & L_{i_1} + L_{j_s} \equiv P_1 + N_s \geqq 0 \\
L_{i_2} + L_{j_1} \equiv P_2 + N_1 \geqq 0, & \ldots, & L_{i_2} + L_{j_s} \equiv P_2 + N_s \geqq 0 \\
\ldots \quad \ldots & \ldots & \ldots \quad \ldots \\
L_{i_r} + L_{j_1} \equiv P_r + N_1 \geqq 0, & \ldots, & L_{i_r} + L_{j_s} \equiv P_r + N_s \geqq 0 \\
L_{k_1} \equiv Z_1 \geqq 0, & \ldots, & L_{k_t} \equiv Z_t \geqq 0.
\end{array} \tag{5}$$

For each solution $(u_1, \ldots, u_{n-1}, \Phi)$ of (5) we must have

$$\Phi \geqq 0. \tag{5'}$$

For, let there be a solution with $\Phi < 0$. Then every P is positive and we may suppose, for convenience, that $P_1 > 0$ is the smallest P. Putting $\Phi = -P_1$ we still have a solution of (5). But by substituting $P_1 = -\Phi$ in the second line of (5) we see that the latter solution is also a solution of (4), which is impossible by (4'). Hence (5') is a consequence of (5).

We now consider the system of inequalities

$$
(6) \qquad
\begin{array}{ccc}
P_1 \geqq 0, & \ldots, & P_r \geqq 0 \\
P_1 + N_1 \geqq 0, & \ldots, & P_1 + N_s \geqq 0 \\
\ldots & \ldots & \ldots \\
P_r + N_1 \geqq 0, & \ldots, & P_r + N_s \geqq 0 \\
Z_1 \geqq 0, & \ldots, & Z_t \geqq 0.
\end{array}
$$

From the assumption made above that no P is identically zero we see that the system (6) contains at least one form which is not identically zero. If (6) has a non-trivial solution then some P must vanish for every solution. For, if this were not the case then for every P_i there would be a solution $u_1^{(i)}, \ldots, u_{n-1}^{(i)}$ of (6) for which $P_i > 0$. The solution $(u_1, \ldots, u_{n-1}) = (u_1^{(1)} + u_1^{(2)} + \ldots, \ldots, u_{n-1}^{(1)} + u_{n-1}^{(2)} + \ldots)$ makes $P_i > 0$ for every $i = 1,2,\ldots,r$. From this we deduce that (5) has a solution with $\Phi < 0$, which is a contradiction of the fact that (5') is a consequence of (5). Hence we may suppose that $P_1 = 0$ for every solution of (6). It follows that

$$
(6') \qquad -P_1 \geqq 0
$$

is a consequence of (6). In the case that (6) has no non-trivial solution then (6') is still a consequence of (6). By our induction assumption there exist non-negative constants a, b, c such that

$$
\begin{aligned}
-P_1 \equiv\ & a_1P_1 + \ldots + a_rP_r + c_1Z_1 + \ldots + c_tZ_t \\
& + b_{11}(P_1 + N_1) + b_{12}(P_1 + N_2) + \ldots + b_{1s}(P_1 + N_s) \\
& \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\
& + b_{r1}(P_r + N_1) + b_{r2}(P_r + N_2) + \ldots + b_{rs}(P_r + N_s).
\end{aligned}
$$

Employing the identities in (5) we find

$$(1+a_1+\dots+a_r)\Phi \equiv (1+a_1)L_{i_1}+\dots+a_rL_r + c_1L_{k_1}+\dots+c_tL_{k_t}$$
$$+ b_{11}(L_{i_1}+L_{j_1}) + b_{12}(L_{i_1}+L_{j_2}) + \dots + b_{1s}(L_{i_1}+L_{j_s})$$
$$+ b_{r1}(L_{i_r}+L_{j_1}) + \dots + b_{rs}(L_{i_r}+L_{j_s}),$$

which proves the theorem.

We define $u = (u_1, u_2, \dots, u_n)$ as a solution of the system

(7) $$A_{\alpha i}u_i >' 0$$

in case $A_{\alpha i}u_i \geqq 0$ is satisfied with the strict inequality holding for at least one value of α. A set of numbers will be called <u>positive definite</u> in case every number of the set is positive.

LEMMA 2:1. <u>A necessary and sufficient condition that</u> (7) <u>admit no solution</u> u <u>is that the system of equalities</u>

(8) $$A_{\alpha i}v_\alpha = 0 \qquad (i = 1,2,\dots,n),$$

<u>admit a positive definite solution</u> $v = (v_1, v_2, \dots, v_m)$.

This is Theorem 12 of Dines and McCoy [3]. We employ this lemma to obtain the following modification of Theorem 2:1.

THEOREM 2:2. <u>If for every non-trivial solution</u> u <u>of</u> (2) <u>it is true that</u> $\Phi \equiv A_i u_i > 0$ <u>then there exist constants</u> $C_\alpha > 0$ <u>such that</u>

$$\Phi \equiv C_1L_1 + C_2L_2 + \dots + C_mL_m.$$

<u>If the matrix</u> $\| A_{\alpha i} \|$ <u>has rank</u> n <u>then the converse is also true.</u>

To prove the first part of the theorem we note that the system of inequalities

$$A_{11}u_1 + A_{12}u_2 + \dots + A_{1n}u_n >' 0$$

$$\dots \qquad \dots \qquad \dots$$

$$A_{m1}u_1 + A_{m2}u_2 + \dots + A_{mn}u_n >' 0$$

$$- A_1u_1 - A_2u_2 - \dots - A_nu_n >' 0$$

has no solution u. We use Lemma 2:1 with (7) replaced by this system and obtain positive constants C_α such that

$$C_\alpha A_{\alpha i} - A_i = 0 \qquad (i = 1,2,\dots,n),$$

as desired. If $\|A_{\alpha i}\|$ has rank n then every non-trivial solution u of $A_{\alpha i}u_i \geqq 0$ is also a solution of $A_{\alpha i}u_i >' 0$. Hence $\phi \equiv C_\alpha L_\alpha > 0$.

For simplicity we use the letter U to denote the class of all non-trivial solutions u of (2).

THEOREM 2:3. The statements:

(i) there exists a $\bar{u}$ satisfying $A_{\alpha i}\bar{u}_i > 0$ for every α,

(ii) U is n-dimensional,

(iii) U is not null and no linear form $A_{\alpha i}u_i$ vanishes for all u belonging to U,

are all equivalent.

The first statement (i) implies (ii) for, by continuity, there is an n-dimensional neighborhood of $\bar{u}$ which belongs to U. The statement (iii) follows from (ii) since if we suppose, for example, that $A_{1i}u_i = 0$ for all u belonging to U then obviously U could not contain n linearly independent vectors u and hence could not be n-dimensional. To prove (iii) implies (i) we notice that there are solutions $u^{(1)}, u^{(2)}, \dots, u^{(m)}$ of (2) such that

$$A_{1i}u_i^{(1)} > 0$$
$$A_{2i}u_i^{(2)} > 0$$
$$\cdots \quad \cdots \quad \cdots$$
$$A_{mi}u_i^{(m)} > 0.$$

Hence we need only set $\bar{u}_i = u_i^{(1)} + u_i^{(2)} + \ldots + u_i^{(m)}$.

For the next theorem we need to introduce the notion of <u>I-rank</u> of a matrix, an integral valued function of a matrix analogous to ordinary rank. But first some preliminary remarks are necessary. Suppose

$$\mathcal{M} = \begin{Vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{Vmatrix}$$

is an $m \times n$ matrix whose elements A_{ij} are all real. The matrix $\mathcal{M}$ is said to be <u>I-definite with respect to a given column</u> in case the elements of that column are all positive, or are all negative; $\mathcal{M}$ will be called <u>I-definite</u> in case it contains at least one column with respect to which it is I-definite.

If $\mathcal{M}$ is not I-definite with respect to the qth column we divide the elements of that column into 3 classes,

r positive elements : A_{iq} $(i = i_1, i_2, \ldots, i_r)$,

s negative elements : A_{jq} $(j = j_1, j_2, \ldots, j_s)$,

t zero elements : A_{kq} $(k = k_1, k_2, \ldots, k_t)$.

From $\mathcal{M}$ we derive the matrix $\mathcal{M}_1^{(q)}$ as follows:

To each pair of elements A_{iq}, A_{jq}, the first positive and the second negative, corresponds one row of $\mathcal{M}_1^{(q)}$ given by

$$\begin{vmatrix} A_{iq} & A_{i1} \\ A_{jq} & A_{j1} \end{vmatrix}, \ldots \begin{vmatrix} A_{iq} & A_{iq-1} \\ A_{jq} & A_{jq-1} \end{vmatrix}, \begin{vmatrix} A_{iq} & A_{iq+1} \\ A_{jq} & A_{jq+1} \end{vmatrix}, \ldots \begin{vmatrix} A_{iq} & A_{in} \\ A_{jq} & A_{jn} \end{vmatrix}.$$

To each zero element A_{kq} corresponds the row

$$A_{k1}, A_{k2}, \ldots, A_{kq-1}, A_{kq+1}, \ldots, A_{kn}.$$

The matrix $\mathfrak{M}_1^{(q)}$ will consist of the rows so formed, the number of rows being $rs + t$. The order of the rows shall be fixed by the rule: (1) each row corresponding to a pair A_{iq}, A_{jq} shall precede every A_{kq} row; (2) of two A_{iq}, A_{jq} rows that one shall precede which has the smaller i or (in case the i's are equal) that one which has the smaller j; (3) of two A_{kq} rows that one shall precede which has the smaller k.

Thus $\mathfrak{M}_1^{(q)}$ is well-defined if $\mathfrak{M}$ is not I-definite with respect to its qth column. If $\mathfrak{M}$ is I-definite with respect to its qth column we define $\mathfrak{M}_1^{(q)}$ as the matrix of 1 row and $(n-1)$ columns all of whose elements are $+1$ or -1 according as the elements of the qth column of $\mathfrak{M}$ are all positive or all negative. The matrix $\mathfrak{M}_1^{(q)}$ will be called the <u>I-complement</u> of the qth column of $\mathfrak{M}$, and the set $\mathfrak{J}_1$ of matrices $\mathfrak{M}_1^{(1)}$, $\mathfrak{M}_1^{(2)}$, ..., $\mathfrak{M}_1^{(n)}$ will be called the <u>I-minors of</u> $(n-1)$ <u>columns</u> of the matrix $\mathfrak{M}$. We notice that if a matrix is I-definite then all its I-complements are likewise I-definite.

Now we form the I-complements for each matrix $\mathfrak{M}_1^{(q)}$, and call the set $\mathfrak{J}_2$ of all such I-complements the I-minors of $(n-2)$ columns of $\mathfrak{M}$. Continuing this process we obtain a finite sequence of sets $\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_{n-1}$ where each matrix in $\mathfrak{J}_p$ is an I-minor of $n-p$ columns of $\mathfrak{M}$. If we define $\mathfrak{M}$ as its own I-minor of n columns then $\mathfrak{J}_0 \equiv \mathfrak{M}$ and the set $\mathfrak{J}_0 + \mathfrak{J}_1 + \cdots + \mathfrak{J}_{n-1}$ of matrices constitute all the I-minors of $\mathfrak{M}$.

We are ready to make the definition: A matrix will be said to be of I-rank h if it possesses at least one I-minor of h columns which is I-definite, but does not possess any I-minor of h + 1 columns which is I-definite. If none of its I-minors are I-definite then it will be said to be of I-rank 0.

In his paper [2] Dines proves the following theorem, the proof of which we shall omit.

THEOREM 2:4. *A necessary and sufficient condition for the existence of a solution $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ of*

$$A_{11}u_1 + A_{12}u_2 + \ldots + A_{1n}u_n > 0$$
$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$
$$A_{m1}u_1 + A_{m2}u_2 + \ldots + A_{mn}u_n > 0$$

is that the I-rank of $\| A_{\alpha i} \|$ be greater than zero.

3. Necessary conditions involving only first derivatives.

We make some preliminary definitions. A solution $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of

$$g_{\alpha x_i}(x^0)\,\lambda_i \geqq 0 \qquad (\alpha = 1,2,\ldots,m),$$

will be called an admissible direction if λ is not the zero vector. A regular arc $x_i(t)$ $(i = 1,2,\ldots,n;\ 0 \leqq t \leqq t_0)$, will be called admissible in case $g_\alpha[x(t)] \geqq 0$ for every α and t. A point x^0 is a normal point in case the matrix

$$\| g_{\alpha x_i}(x^0) \|$$

has rank m.

THEOREM 3:1. *If $f(x^0)$ is a minimum then there exist multipliers χ_0, χ_α not all zero such that the derivatives F_{x_i} of the function*

$$F(x) = \lambda_0 f(x) + \lambda_\alpha g_\alpha(x)$$

<u>all</u> <u>vanish</u> <u>at</u> x^o.

In the class of points $(x,z) = (x_1,\ldots,x_n,z_1,\ldots,z_m)$ satisfying $h_\alpha(x,z) = g_\alpha(x) - z_\alpha^2 = 0$ the point $(x,z) = (x^o,0)$ is a minimizing point for f. Hence, by Theorem 1:1, there exist constants λ_0, λ_α not all zero such that the function $H(x,z) = \lambda_0 f + \lambda_\alpha h_\alpha = \lambda_0 f + \lambda_\alpha g_\alpha - \lambda_\alpha z_\alpha^2$ has $H_{x_i}(x^o,0) = 0$. It follows that $F(x) = \lambda_0 f + \lambda_\alpha g_\alpha$ has $F_{x_i}(x^o) = 0$.

We note that if $m < n$ the above proof of Theorem 3:1 is unnecessary. For, if x^o is a minimizing point in the class of points satisfying $g_\alpha(x) \geqq 0$ it certainly is a minimizing point in the class satisfying $g_\alpha(x) = 0$, and Theorem 1:1 can be applied directly.

If x^o is a minimizing point for f which is normal then the multiplier λ_0 is not zero and can be given the value one by dividing each number of the set λ_0, λ_α by λ_0 and obtaining a new set $\lambda_0 = 1$, λ_α which satisfies the conclusion of the theorem. Assume, then, that x^o is a normal minimizing point and $\lambda_0 = 1$. If we suppose, for the moment, that the functions f and g_α have continuous second derivatives then by employing the necessary condition on the second derivatives of $H(x,z)$ at the minimizing point $(x,z) = (x^o,0)$ of f as given in Theorem 1:2 we can easily show that $\lambda_\alpha \leqq 0$ $(\alpha = 1,2,\ldots,m)$. For, by the theorem just referred to, the quadratic form

$$\eta_i \eta_k H_{x_i x_k}(x^o,0) + 2\eta_i \zeta_\alpha H_{x_i z_\alpha}(x^o,0) + \zeta_\alpha \zeta_\beta H_{z_\alpha z_\beta}(x^o,0) =$$

$$\eta_i \eta_k F_{x_i x_k}(x^o) - 2\zeta_\alpha^2 \lambda_\alpha$$

must be non-negative for all sets $\eta_1, \eta_2, \ldots, \eta_n, \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$ for which $\eta_i g_{\alpha x_i}(x^o) = 0$ and $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$ is arbitrary. Setting every η and Υ except Υ_α equal to zero and substituting in the quadratic form we find that $\chi_\alpha \leqq 0$.

However, in this section we shall make proofs of the non-positive character of the multipliers χ_α which do not involve second derivatives, and the case when the minimizing point is normal will appear as a special instance (see the proof of the corollary to Theorem 5:1).

We use Theorem 2:1 to obtain the following necessary condition.

THEOREM 3:2. <u>Suppose that for each admissible direction λ there is an admissible arc issuing from x^o in the direction λ. Then a first necessary condition for $f(x^o)$ to be a minimum is that there exist multipliers $\chi_\alpha \leqq 0$ such that the derivatives F_{x_i} of the function</u>

$$F = f + \chi_\alpha g_\alpha$$

<u>all vanish at x^o.</u>

By a curve $x_i(t)$ $(0 \leqq t \leqq t_o)$, "issuing from x^o in the direction λ" we mean, of course, that $x_i(0) = x_i^o$ and $x_i'(0) = \lambda_i$. Consider an admissible direction λ and the corresponding admissible curve $x_i(t)$ given in the hypothesis. Let $\mathcal{F}(t) = f[x(t)]$. Since $\mathcal{F}(0) \leqq \mathcal{F}(t)$ for $0 \leqq t \leqq t_o$, it follows that $\mathcal{F}'(0) \geqq 0$. But $\mathcal{F}'(0) = f_{x_i}(x^o)\lambda_i$. Hence $f_{x_i}(x^o)\lambda_i \geqq 0$. Then $f_{x_i}(x^o)u_i \geqq 0$ is a consequence of $g_{\alpha x_i}(x^o)u_i \geqq 0$ $(\alpha = 1,2,\ldots,m)$, and by Theorem 2:1 there exist multipliers $\chi_\alpha \leqq 0$ such that $f_{x_i}(x^o)u_i + \chi_\alpha g_{\alpha x_i}(x^o)u_i \equiv 0$. Thus $f_{x_i} + \chi_\alpha g_{\alpha x_i} = 0$ for every i, and the theorem is proved.

The condition that there exist multipliers $\chi_\alpha \leqq 0$ satisfying the conclusion of Theorem 3:2 will be referred to as "the first necessary condition". For brevity, the property that for each admissible direction λ there is an admissible arc issuing from x^o in the direction λ will be called <u>property Q</u>.

One would naturally like to know what the probability is, roughly, that the functions $g_\alpha(x)$ will satisfy property Q, as well as some conditions on the functions g_α which will ensure the satisfaction of Q. In order to partially answer these questions we shall briefly discuss one geometric interpretation of an admissible direction.

The tangent planes to the surfaces $g_\alpha(x) = 0$ at their common point of intersection x^o are given by

$$T_\alpha(x) \equiv g_{\alpha x_i}(x^o)(x_i - x_i^o) = 0 \qquad (\alpha = 1,2,\ldots,m).$$

The straight line issuing from x^o in the admissible direction λ is

$$\text{(9)} \qquad S:\ x_i(t) = \lambda_i t + x_i^o \qquad (0 \leqq t \leqq t_o;\ i = 1,2,\ldots,n).$$

Substituting the equations of S in $T_\alpha(x)$ we obtain $T_\alpha[x(t)] \geqq 0$. We conclude that the line S lies in the set of points x near x^o satisfying $T_\alpha(x) \geqq 0$; and since the latter set, in a sense, approximates the set of points x near x^o satisfying $g_\alpha(x) \geqq 0$, if the functions g_α are regular enough, it seems that the satisfaction of property Q is not a great restriction on the functions g_α. In fact, the following corollary states a condition on g_α which makes the line S an admissible arc.

COROLLARY. <u>Suppose that for every admissible direction</u> λ <u>it is true that</u> $g_{\alpha x_i}(x^o)\lambda_i = 0$ <u>implies that</u> $g_{\alpha x_i x_k}(x^o)\lambda_i\lambda_k > 0$.

<u>Then</u> <u>if</u> $f(x^o)$ = <u>minimum</u> <u>the</u> <u>first</u> <u>necessary</u> <u>condition</u> <u>is</u> <u>satisfied</u>.

Consider any admissible direction λ and the corresponding line S given in (9). Define $\bar{g}_\alpha(t) = g_\alpha[x(t)]$ $(\alpha = 1,2,\ldots,m;$ $0 \leqq t \leqq t_0)$. We have $d\bar{g}_\alpha(t)/dt = g_{\alpha x_i}[x(t)]x_i{}'(t) = g_{\alpha x_i}[x(t)]\,\lambda_i$. Hence

$$\frac{d\bar{g}_\alpha(0)}{dt} = g_{\alpha x_i}(x^o)\ \lambda_i \geqq 0.$$

If $d\bar{g}_\alpha(0)/dt > 0$ then $\bar{g}_\alpha(t)$ is monotonically increasing near $t = 0$ and $\bar{g}_\alpha(t) = g_\alpha[x(t)] \geqq g_\alpha(x^o) = 0$. Hence S lies in the set of points x satisfying $g_\alpha(x) \geqq 0$. If $d\bar{g}_\alpha(0)/dt = 0$ then $d^2\bar{g}_\alpha(t)/dt^2 = g_{\alpha x_i x_k}[x(t)]\ \lambda_i\ \lambda_k$ and by hypothesis

$$\frac{d^2\bar{g}_\alpha(0)}{dt^2} = g_{\alpha x_i x_k}(x^o)\ \lambda_i\ \lambda_k > 0.$$

Therefore $\bar{g}_\alpha(t)$ is monotonically increasing and, as before, satisfies $g_\alpha[x(t)] \geqq 0$. We have shown that with S the hypotheses of Theorem 3:2 are satisfied, and the conclusion follows.

In Theorem 3:3 we obtain the same necessary condition that Theorem 3:2 yielded but under a different hypothesis.

THEOREM 3:3. <u>Suppose</u> <u>there</u> <u>exists</u> <u>an</u> <u>admissible</u> <u>direction</u> $\bar{\lambda}$ <u>for</u> <u>which</u> $g_{\alpha x_i}(x^o)\ \bar{\lambda}_i > 0$ <u>for</u> <u>every</u> α. <u>Then</u> <u>if</u> $f(x^o)$ = <u>minimum</u> <u>the</u> <u>first</u> <u>necessary</u> <u>condition</u> <u>is</u> <u>satisfied</u>.

First we prove that if λ is such that $g_{\alpha x_i}(x^o)\ \lambda_i > 0$ for every α then $f_{x_i}(x^o)\ \lambda_i \geqq 0$. Let g represent any one of the g_α and define, as before,

$$\bar{g}(t) = g[x(t)], \qquad \bar{f}(t) = f[x(t)],$$

where x(t) represents the equations of the line S in (9). Since $d\bar{g}(0)/dt = g_{x_i}(x^o)\ \lambda_i > 0$, $\bar{g}(t)$ is monotonically increasing, $g[x(t)] \geqq g(x^o) = 0$, and S is an admissible arc. Thus $\bar{f}(0) \leqq \bar{f}(t)$, and consequently

$$\frac{dF(0)}{dt} = f_{x_i}(x^o)\,\lambda_i \geqq 0.$$

Now suppose μ_i is an admissible direction. We define a family of directions

$$\nu_i(s) = \bar{\lambda}_i + s(\mu_i - \bar{\lambda}_i) \qquad (0 \leqq s \leqq 1),$$

where $\bar{\lambda}$ is given in the hypothesis of the theorem. Rewriting $\nu_i(s) = (1-s)\bar{\lambda}_i + s\mu_i$, it is clear that $g_{x_i}(x^o)\,\nu_i(s) > 0$ for $0 \leqq s < 1$. From the first part of the proof,

$$f_{x_i}(x^o)\,\nu_i(s) \geqq 0 \qquad (0 \leqq s < 1),$$

so that

$$\lim_{s\to 1} f_{x_i}(x^o)\,\nu_i(s) = f_{x_i}(x^o)\,\mu_i \geqq 0.$$

Hence the inequality $f_{x_i}(x^o)\,u_i \geqq 0$ is a consequence of $g_{\alpha x_i}(x^o)\,u_i \geqq 0$ $(\alpha = 1,2,\ldots,m)$, and the theorem follows from Theorem 2:1.

Suppose $m = n$ and the determinant of $\| g_{\alpha x_i}(x^o) \|$ is different from zero. For this case we can write the first necessary condition in an entirely equivalent form as follows.

COROLLARY. *Suppose* $m = n$ *and determinant* $\| g_{\alpha x_i}(x^o) \| \neq 0$. *Then a necessary condition for* $f(x^o)$ *to be a minimum is that*

$$f_{x_i}(x^o) G_{i\alpha} \geqq 0 \qquad (\alpha = 1,2,\ldots,n),$$

where $\| G_{i\alpha} \|$ *is the inverse matrix of* $\| g_{\alpha x_i}(x^o) \|$.

The system of equations

$$g_{1x_i}(x^0)u_i = 1$$
$$\cdots \qquad \cdots$$
$$g_{nx_i}(x^0)u_i = 1$$

has a solution $u = \bar{\lambda}$ since determinant $\| g_{\alpha x_i}(x^0) \| \neq 0$. Thus

$$g_{\alpha x_i}(x^0)\bar{\lambda}_i > 0,$$

and we can apply Theorem 3:3 to obtain the first necessary condition; that is, there exist multipliers $\gamma_\alpha \leqq 0$ such that

$$f_{x_i}(x^0) = -\gamma_\alpha g_{\alpha x_i}(x^0).$$

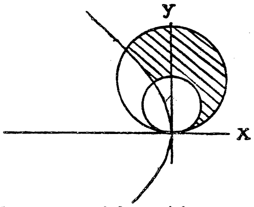
Multiplying both sides of the last equation by $G_{i\beta}$ and summing with respect to the index i, we obtain

$$f_{x_i}(x^0)G_{i\beta} = -\gamma_\beta \geqq 0,$$

as desired.

The problem of determining necessary and sufficient conditions for the existence of an admissible direction λ satisfying $g_{\alpha x_i}(x^0)\bar{\lambda}_i > 0$ naturally arises in the consideration of Theorem 3:3. The question is answered by Theorems 2:3 and 2:4. In particular, the latter theorem provides a useful method for determining in a finite number of steps whether or not such an admissible vector λ does exist.

It is easy to give an example in which the functions g_α satisfy neither the hypothesis of the corollary to Theorem 3:2 nor the hypothesis of Theorem 3:3, but in which the hypothesis of Theorem 3:2 is satisfied. Let



$$g_1(x,y) = x^2 + (y-1)^2 - 1 \geqq 0$$
$$g_2(x,y) = 4 - [x^2 + (y-2)^2] \geqq 0$$
$$g_3(x,y) = y^2 + x \geqq 0$$

determine the class of points (x,y) under consideration. At (0,0) we have

$$\begin{Vmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \\ g_{3x} & g_{3y} \end{Vmatrix} = \begin{Vmatrix} 0 & -2 \\ 0 & 4 \\ 1 & 0 \end{Vmatrix}$$

The only admissible direction is (a,0) with $a > 0$. There is no solution of $g_{\alpha x}(0,0)\lambda_1 + g_{\alpha y}(0,0)\lambda_2 > 0$ for all α. Also $g_{2xx}(0,0)a^2 < 0$ so that the hypothesis of the corollary to Theorem 3:2 is not satisfied. However, it is obvious that there is an admissible arc issuing from (0,0) in the direction (a,0).

4. Sufficient conditions involving only first derivatives. By a proper strengthening of the first necessary condition we can obtain a sufficiency theorem without resorting to second derivatives.

THEOREM 4:1. *Suppose* $m \geqq n$ *and* $\| g_{\alpha x_i}(x^0) \|$ *has maximum rank* n. *If* x^0 *is a point satisfying* $g_\alpha(x^0) = 0$ *for which there exist multipliers* $\chi_\alpha < 0$ *such that* $F = f + \chi_\alpha g_\alpha$ *has* $F_{x_i}(x^0) = 0$, *then* $f(x^0)$ *is a minimum.*

By Taylor's expansion formula,

$$f(x) - f(x^0) = f_{x_i}(x')\eta_i$$
$$0 \leqq g_\alpha(x) = g_{\alpha x_i}(x_\alpha')\eta_i \qquad (\alpha = 1,2,\ldots,m),$$

for x near x^0 and x satisfying $g_\alpha(x) \geqq 0$, where $\eta_i = x_i - x_i^0$, $x'_{\alpha i} = x_i^0 + \theta_\alpha(x_i - x_i^0)$. By hypothesis,

$$
(10) \qquad \begin{aligned}
c_1 g_{1x_1}(x^0) + c_2 g_{2x_1}(x^0) + \dots + c_m g_{mx_1}(x^0) &= f_{x_1}(x^0) \\
\dots \qquad\qquad \dots \qquad\qquad &\dots \\
c_1 g_{1x_n}(x^0) + c_2 g_{2x_n}(x^0) + \dots + c_m g_{mx_n}(x^0) &= f_{x_n}(x^0)
\end{aligned}
$$

where $c_\alpha = -\gamma_\alpha > 0$. For convenience suppose

$$
\begin{vmatrix}
g_{1x_1}(x^0) & \dots & g_{nx_1}(x^0) \\
\dots & \dots & \dots \\
g_{1x_n}(x^0) & \dots & g_{nx_n}(x^0)
\end{vmatrix} \neq 0.
$$

We fix $c_{n+1},\dots,c_m$ in (10) and solve for $c_1,\dots,c_n$ as continuous functions of the coefficients $g_{\alpha x_i}(x^0)$ and $f_{x_i}(x^0)$. Hence for $A_{\alpha i}$ sufficiently close to $g_{\alpha x_i}(x^0)$ and A_i sufficiently close to $f_{x_i}(x^0)$ there exists a unique solution $\bar{c}_1 > 0$, $\bar{c}_2 > 0$, ..., $\bar{c}_m > 0$ of

$$
\bar{c}_\alpha A_{\alpha i} = A_i \qquad\qquad (i = 1,2,\dots,n).
$$

Hence for x sufficiently close to x^0 there exist constants $\bar{c}_1 > 0$, ..., $\bar{c}_m > 0$ such that

$$
\bar{c}_\alpha g_{\alpha x_i}(x_\alpha') = f_{x_i}(x'),
$$

$$
f(x) - f(x^0) = f_{x_i}(x')\eta_i = \bar{c}_\alpha g_{\alpha x_i}(x_\alpha')\eta_i \geqq 0,
$$

and $f(x^0)$ is a minimum.

We have a sufficiency theorem corresponding to the necessary condition in the corollary to Theorem 3:3.

COROLLARY. *Suppose $m = n$ and determinant $\| g_{\alpha x_i}(x^0) \| \neq 0$. We let $\| G_{i\alpha} \|$ be the inverse matrix of $\| g_{\alpha x_i} \|$. If x^0 is a point satisfying $g_\alpha(x^0) = 0$ such that*

$$
f_{x_i}(x^0) G_{i\alpha} > 0 \qquad\qquad (\alpha = 1,2,\dots,n),
$$

then $f(x^o)$ *is a minimum*.

We define $\chi_\alpha < 0$ by the equation

$$f_{x_i}(x^o)G_{i\alpha} = -\chi_\alpha .$$

Multiplying both sides by $g_{\alpha x_j}(x^o)$ and summing with respect to the index α, we obtain

$$f_{x_j}(x^o) = -\chi_\alpha g_{\alpha x_j}(x^o),$$

and the conclusion follows from Theorem 4:1.

The following sufficiency theorem is entirely equivalent to Theorem 4:1.

THEOREM 4:2. *Suppose* $m \geqq n$ *and* $\| g_{\alpha x_i}(x^o)\|$ *has rank* n. *If* x^o *is a point satisfying* $g_\alpha(x^o) = 0$ *such that* $f_{x_i}(x^o)\lambda_i > 0$ *for every admissible direction* λ, *then* $f(x^o)$ *is a minimum*.

This result follows at once from Theorem 2:2 and Theorem 4:1.

5. *A necessary condition involving second derivatives*. Suppose $f(x^o)$ is a minimum, $\| g_{\alpha x_i}(x^o)\|$ has rank r, and for convenience the first r row vectors are linearly independent. We also suppose that there exist multipliers χ_α such that $F = f + \chi_\alpha g_\alpha$ has $F_{x_i}(x^o) = 0$, that is,

$$f_{x_i}(x^o) + \chi_\alpha g_{\alpha x_i}(x^o) = 0 \qquad (i = 1,2,\ldots.n).$$

Since all the row vectors are linear combinations of the first r we may suppose $\chi_{r+1} = 0, \ldots, \chi_m = 0$. In this form the multipliers χ_α are unique for, if χ'_α is any other set with $\chi'_{r+1} = 0, \ldots, \chi'_m = 0$ then $\sum_{\alpha=1}^{r} (\chi'_\alpha - \chi_\alpha)g_{\alpha x_i}(x^o) = 0$ and hence $\chi'_\alpha = \chi_\alpha$ $(\alpha = 1,2,\ldots,r)$.

If the hypotheses of Theorem 3:3 are satisfied and $\| g_{\alpha x_i}(x^0) \|$ has rank $r = 1$ or 2 we can show that there are respectively one or two linearly independent rows whose unique multipliers are non-positive. It is obviously sufficient to prove the following proposition: If there exists an admissible direction $\bar{\lambda}$ satisfying $g_{\alpha x_i}(x^0)\bar{\lambda}_i > 0$, every row of $\| g_{\alpha x_i}(x^0) \|$ is a linear combination with non-negative coefficients of some r linearly independent rows ($r = 1$ or 2). If $r = 1$ the proof is obvious. If $r = 2$ we make an inductive proof. The case $m = 2$ is clear. We assume the proposition for $m - 1$ and make the proof for m. By our induction assumption we may suppose that the first two rows of $\| g_{\alpha x_i}(x^0) \|$ are linearly independent and every other row, except possibly the last, is a linear combination with non-negative coefficients of these two. For the last row we have

$$a g_{1x_i} + b g_{2x_i} + c g_{mx_i} = 0 \qquad (i = 1,2,\ldots,n),$$

with $(a,b,c) \neq (0,0,0)$. Hence $a g_{1x_i}\bar{\lambda}_i + b g_{2x_i}\bar{\lambda}_i + c g_{mx_i}\bar{\lambda}_i = 0$. The numbers a, b, c cannot all be of the same sign. For, if they were then the last expression would be different from zero since $g_{\alpha x_i}\bar{\lambda}_i > 0$. Hence one of the three vectors is a linear combination with non-negative coefficients of the other two, and it follows that every row vector is a linear combination with non-negative coefficients of the same two.

That one cannot hope to extend the above proposition to the case when $r \geqq 3$ is shown by the following example. Let

$$\| g_{\alpha x_i}(x^0) \| = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \end{Vmatrix}$$

The system of inequalities

$$g_{\alpha x_i}(x^o)\,\lambda_i > 0$$

has a solution $(\bar\lambda_1, \bar\lambda_2, \bar\lambda_3, \bar\lambda_4) = (1,1,1,1)$. If we take the linear combination of the rows with respective coefficients -1, -1, +1, +1 we obtain the zero vector. Since the rank of $\| g_{\alpha x_i}(x^o)\|$ is three any solution v of $g_{\alpha x_i} v_\alpha = 0$ is given by $v = k(-1,-1,+1,+1)$. Hence no row can be a linear combination with positive coefficients of the other three rows.

The next theorem gives a necessary condition involving the second derivatives of the functions f and g_α.

THEOREM 5:1. _Suppose_ $f(x^o)$ _is a minimum and there exist multipliers_ χ_α _such that_ $\bar F = f + \chi_\alpha g_\alpha$ _has_ $\bar F_{x_i}(x^o) = 0$. _Suppose, further, that_ $\| g_{\alpha x_i}(x^o)\|$ _has rank_ $r < n$ _with the first_ r _rows linearly independent. Then for every admissible direction_ η _satisfying_ $g_{\alpha x_i}(x^o)\,\eta_i = 0 \quad (\alpha = 1,2,\ldots,m)$, _such that there is an admissible arc_ $x(t)$ _of class_ C'' _issuing from_ x^o _in the direction_ η _and satisfying_ $g_\alpha[x(t)] = 0$ _for_ $\alpha = 1,2,\ldots,r$, _it is true that_

$$F_{x_i x_k}(x^o)\,\eta_i\,\eta_k \geqq 0,$$

where F _is formed with the unique set of multipliers_ χ_α _belonging to the first_ r _rows of_ $\| g_{\alpha x_i}(x^o)\|$.

We notice that for any particular η satisfying $g_{\alpha x_i}(x^o)\eta_i = 0$ the selection of the r linearly independent rows that shall satisfy with η the hypotheses of the theorem, depends upon η. In the statement of the theorem we have taken an η and renumbered the functions g_α so that the r linearly independent rows going with η are the first r rows.

We have $g_\alpha[x(t)] \equiv 0$ and hence $g_{\alpha x_i} x_i'(t) \equiv 0$ for $\alpha = 1,2,\ldots,r$. Let $\bar{f}(t) = f[x(t)]$. Then

$$'(t) = f_{x_i}[x(t)]x_i'(t) = (f_{x_i} + \sum_{\alpha=1}^{r} \lambda_\alpha g_{\alpha x_i})x_i'(t) = F_{x_i}[x(t)]x_i'(t),$$

$$\bar{f}'(0) = F_{x_i}[x^o]\eta_i = 0.$$

ut since $f(x^o)$ is a minimum $\bar{f}(0) \leqq \bar{f}(t)$, $\bar{f}''(0) = F_{x_i x_k}[x^o]\eta_i\eta_k \geqq 0$, nd the theorem is proved.

Theorem 5:1 can be applied in the particular case when $x^o)$ is a normal point.

COROLLARY. <u>Suppose x^o is a normal point. Then necessary onditions for $f(x^o)$ to be a minimum are that the first necessary ondition be satisfied and that</u>

$$F_{x_i x_k}(x^o)\eta_i\eta_k \geqq 0$$

<u>e satisfied for every admissible direction η satisfying</u>

$$g_{\alpha x_i}(x^o)\eta_i = 0 \qquad (\alpha = 1,2,\ldots,m).$$

The first necessary condition is easily proved by means of heorem 3:3. For, since the rank of $\| g_{\alpha x_i} \|$ is m there exists solution $\bar{\lambda}$ of $g_{\alpha x_i}(x^o)\bar{\lambda}_i = 1$ $(\alpha = 1,2,\ldots,m)$, and hence a olution of $g_{\alpha x_i}(x^o)\bar{\lambda}_i > 0$.

If the rank of $\| g_{\alpha x_i}(x^o) \|$ is $m = n$ then the second ecessary condition in the corollary is vacuously satisfied since o η exists for which $g_{\alpha x_i}(x^o)\eta_i = 0$. If $m < n$ the second ecessary condition follows if we notice that Lemma 1:1 enables s to satisfy the hypotheses of Theorem 5:1.

6. <u>A sufficiency theorem involving second derivatives.</u> orresponding to Theorem 1:3 we have the following sufficiency heorem.

THEOREM 6:1. *If a point* x^o *satisfying* $g_\alpha(x^o) = 0$ *has a* *et of multipliers* $\lambda_\alpha < 0$ *for which the function* $F = f + \lambda_\alpha g_\alpha$ *atisfies*

$$F_{x_i}(x^o) = 0, \qquad F_{x_i x_k}(x^o)\eta_i \eta_k > 0$$

or all admissible directions η *satisfying*

$$g_{\alpha x_i}(x^o)\eta_i = 0,$$

hen $f(x^o)$ *is a minimum.*

The proof consists of verifying that the hypotheses of heorem 1:3 are satisfied for the problem of showing that $(x,z) =$ $x^o,0)$ is a minimizing point for f in the class of points $(x,z) =$ $x_1,\ldots,x_n,z_1,\ldots,z_m)$ satisfying

$$h_\alpha(x,z) = g_\alpha(x) - z_\alpha^2 = 0 \qquad (\alpha = 1,2,\ldots,m).$$

et $H(x,z) = f + \lambda_\alpha h_\alpha = F(x) - \lambda_\alpha z_\alpha^2$. Then

$$H_{x_i}(x^o,0) = F_{x_i}(x^o) = 0, \qquad H_{z_k} = 0.$$

onsider any set $(\eta_i, \zeta_k) \neq (0,0)$ $(i = 1,2,\ldots,n;\ k = 1,2,\ldots,m)$, uch that

$$h_{\alpha x_i}(x^o,0)\eta_i + h_{\alpha z_k}(x^o,0)\zeta_k = 0 \qquad (\alpha = 1,2,\ldots,m),$$

hat is, such that

$$g_{\alpha x_i}(x^o)\eta_i = 0, \qquad \zeta_k \text{ arbitrary.}$$

e quadratic form formed with the second derivatives of H is

$$H_{x_i x_k}(x^o,0)\eta_i\eta_k + 2H_{x_i z_k}(x^o,0)\eta_i\zeta_k + H_{z_k z_\ell}(x^o,0)\zeta_k\zeta_\ell$$

ich reduces to

$$F_{x_i x_k}(x^0)\eta_i\eta_k - 2\lambda_k \zeta_k^2 > 0.$$

ence $(x^0,0)$ is a minimizing point. It follows that (x^0) is a inimizing point for the original problem.

Under the assumption that the functions f and g_α have ontinuous derivatives of at least the second order, Theorem 4:1 s an immediate corollary of Theorem 6:1. However, as observed efore, Theorem 4:1 also holds for the case when f and g_α have ontinuous derivatives of only the first order.

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VITA

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